

THE MOTIVIC COHOMOLOGY OF BSO_n

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ABSTRACT. We will determine the motivic cohomology $H^{*,*}(BSO_n, \mathbb{Z}/2)$ with coefficients in $\mathbb{Z}/2$ of the classifying space of special orthogonal groups SO_n over the complex numbers \mathbb{C} .

0. INTRODUCTION

We denote by BG_k the classifying space of a linear algebraic group G_k over a field k . It is a colimit of smooth schemes defined by B. Totaro [16], and F. Morel and V. Voevodsky in [9, §4], where it is called the geometric model. SO_n is the special orthogonal group of dimension n , and we will treat cases where k is the field of complex numbers \mathbb{C} . By $H^{*,*}(X)$ we mean the bi-graded motivic cohomology group with coefficients in $\mathbb{Z}/2$,

$$\bigoplus_{i,j \geq 0} H^{i,j}(X) = \bigoplus_{i,j \geq 0} H^i(X, \mathbb{Z}/2(j)),$$

of a smooth scheme X . We use a definition used in [19] and [20]. We call the first index i by degree and $2j - i$ by weight. (Total weight may be appropriate. But we simply call this way.) Since we will only consider cohomology with coefficients in $\mathbb{Z}/2$, the coefficients are usually omitted in the notation. The motivic cohomology of BSO_n is defined as a limit of motivic cohomology of smooth schemes.

The motivic cohomology groups are related to the Chow groups and ordinary cohomology groups in the following way. $H^{2i,i}(X)$ is isomorphic to the mod 2 reduction $CH^i(X) \otimes \mathbb{Z}/2$ of the i -th Chow group for a smooth scheme X . In our convention $\bigoplus H^{2i,i}(X)$ is the subgroup of elements having weight 0. There are no elements having negative weights. There exists a realization mapping

$$(1) \quad t_X^{i,j} : H^{i,j}(X, \mathbb{Z}/2) \rightarrow H^i(X, \mathbb{Z}/2),$$

where $t_X^{2i,i}$ is identified to the cycle class mapping. It was called the Beilinson-Lichtenbaum conjecture [19, §6] that it becomes an isomorphism when $i \leq j$, which is proved in [19].

$H^*(BG, \mathbb{Z}/2) = H^*(BG)$ is the singular cohomology group of CW-complex $BG(\mathbb{C})$, which is homotopy equivalent to the classifying space of a maximal compact subgroup of $G(\mathbb{C})$.

When $G = GL_n, SL_n, Sp_n$, the motivic cohomology of BG is known over arbitrary fields integrally. For example $H^{*,*}(BGL_n, \mathbb{Z})$ is equal to

$H^{*,*}(\text{Spec } k, \mathbb{Z})[c_1, \dots, c_n]$ where c_i is the Chern class of the standard representation. N. Yagita determined $H^{*,*}(BSO_4)$ over \mathbb{C} [24], where cases for $G = O_n, SO_{2m+1}, G_2, Spin_7$ and some other groups in different coefficients are treated. It completes his preceding results in [22], [23]. We will recall his argument in 1.3. Note that $SO_2 = GL_1$ for $\sqrt{-1} \in k$. Therefore the new results in this paper are cases $G = SO_{2m}$ for $m \geq 3$.

We know that D. Edidin and W. Graham [1] constructed an element in $CH^m(SO_{2m})$ for each $m \geq 2$ such that the mod 2 reduction of it, $y_{0,m}$ in $H^{2m,m}(BSO_{2m})$, becomes 0 through cycle class mapping

$$t_{BSO_{2m}}^{*,*}(y_{0,m}) = 0.$$

(The entire structure of the group $CH^*(SO_{2m})$ is known by R. Pandharipande [11] in the case $m = 2$ and R. Field [4] [3] in general m .) L. Molinaro-Rojas and A. Vistoli [8] gave a new insight on the element using the stratified method. We will review it in 1.2. This phenomenon is first observed by B. Totaro [15], that a generator in cobordism groups of $BSO_4(\mathbb{C})$ studied in [7] would not map to a generator of ordinary cohomology by his refined cycle class mapping. Note that the cobordism group of $BSO_n(\mathbb{C})$ is determined in [6].

N. Yagita found a series of elements $y_{i,m}$ in $H^{2m,i+m}(BSO_{2m})$ for any i from 0 to $m - 2$ in the the kernel of $t_{BSO_{2m}}^{*,*}$ [24, 9.3], where it was denoted by u_{i+1} . These elements will be defined in 3.3. Since $H^{*,*}(\text{Spec } \mathbb{C}) = \mathbb{Z}/2[\tau]$ ($\tau \in H^{0,1}(\text{Spec } \mathbb{C})$) [19, 7.8], $H^{*,*}(BSO_n)$ is a $\mathbb{Z}/2[\tau]$ -algebra. $\text{Ker}(t_{BSO_{2m}}^{*,*})$ becomes the ideal of τ -torsion elements exactly [14, §6].

Our result is that $\text{Ker } t_{BSO_{2m}}^{*,*}$ is generated by Yagita's elements $y_{i,m}$ over the polynomial ring of Chern classes of even indices; $0 = \tau y_{i,m} = z y_{i,m}$ if z is not a polynomial of c_{2i} 's and $y_{i,m} y_{j,m} = 0$:

Theorem 0.1. *The kernel of realization mapping $\text{Ker } t^{*,*}$ in $H^{*,*}(BSO_{2m})$ is*

$$\text{Ker}(t_{BSO_{2m}}^{*,*}) = \mathbb{Z}/2[c_2, c_4, \dots, c_{2m}]\{y_{0,m}, \dots, y_{m-2,m}\}.$$

Note that $\text{Ker}(t_X^{2m,2m-1}) = 0$ for any X (It is a direct consequence of the theorems of Voevodsky [19, Theorem 6.1, Lemma 6.9, Theorem 7.4]).

0.1. Conventions. Since we only consider coefficients in $\mathbb{Z}/2$, we will omit it. We have $H^{*,*}(\text{Spec } \mathbb{C}) = \mathbb{Z}/2[\tau]$ where τ has degree 0 and weight 2. The motivic cohomology groups $H^{*,*}(X)$ is a bi-graded algebra over $\mathbb{Z}/2[\tau]$.

The group of quadratic roots of unity μ_2 has the motivic cohomology $H^{*,*}(B\mu_2) = \mathbb{Z}/2[\tau, y] \otimes \Lambda(x) = \mathbb{Z}/2[\tau, y]\{1, x\}$ where $x \in H^{1,1}(B\mu_2)$ and $x^2 = \tau y$ [20, 6.10]. Using the Beilinson-Lichtenbaum conjecture, we will identify $H^i(X) = H^{i,i}(X)$. We use the equivariant motivic cohomology in 1.1, which is denoted by $H_G^{*,*}(X)$. We use a simplified notation $H_G^{*,*} = H_G^{*,*}(\text{Spec } \mathbb{C}) = H^{*,*}(BG)$. Similarly $H_G^* = H_G^*(pt)$.

The orthogonal group O_n is the algebraic subgroup of GL_n consisting of elements preserving the quadratic form $q(x) = \sum x_i^2$. The special orthogonal

group SO_n is the subgroup of O_n consisting of elements with determinant one, and the inclusion is denoted $\epsilon : SO_n \subset O_n$. We regard O_{n-1} as the subgroup fixing the last coordinate in O_n . Similarly for $\iota : SO_{n-1} \subset SO_n$. We denote by ι any of those canonical inclusions. We use the same notation for an induced morphism between classifying spaces as $\iota : BSO_{n-1} \subset BSO_n$. $\det : O_n \rightarrow \mu_2 \subset \mathbb{G}_m$ will be the determinant morphism.

Let us denote by γ_n the standard real n -dimensional representation of $O_n(\mathbb{R})$, or its associated bundle on $BO_n(\mathbb{R})$. For $G = O_n, SO_n$, $w_i \in H^i(BG(\mathbb{R})) = H^i(BG(\mathbb{C})) = H^i(BG) = H^{i,i}(BG)$ will mean the i -th Stiefel-Whitney class of γ_n or $\epsilon^{-1}\gamma_n$. It is classical that $H^*(BO_n) = \mathbb{Z}/2[w_1, w_2, \dots, w_n]$ and $H^*(BSO_n) = \mathbb{Z}/2[w_2, \dots, w_n]$. The i -th Chern class c_i of $\gamma \otimes \mathbb{C}$ or ϵ^{-1} -image of it, is an element in $CH^i(BG) \otimes \mathbb{Z}/2 = H^{2i,i}(BG)$ so that $(w_i)^2 = \tau^i c_i$.

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1. PRELIMINARIES

1.0. Motivic cohomology and the filtration by weight. The realization mapping $t = t_X = t_X^{i,j} : H^{i,j}(X) \rightarrow H^{i,i}(X) = H^i(X)$ is just a multiplication of τ^{i-j} [14, §6]. We can introduce a filtration by weight on $H^i(X)$, so that $F^j H^i(X)$ consists of elements which are the realization of weight less than j . It means x has weight exactly $2j - i$ if there exists x' as $x = \tau^{i-j} x'$ and j is the minimum of such numbers. We denote such x' as \bar{x} , which is not determined uniquely if there exists non-trivial element of τ -torsion. But in all the cases appear in our paper we have a unique choice of such element for a homogeneous x by reasoning on bi-degree. It defines a filtered ring structure on $H^i(X)$ [24, §2, §6, §7], which is equivalent to the coniveau filtration of Grothendieck, after shifts on degrees and indices of filtration. The precise argument is in [24, §7].

A morphism $f : X \rightarrow Y$ induces a pullback $f^* : H^{*,*}(Y) \rightarrow H^{*,*}(X)$ that is a homomorphism of $\mathbb{Z}/2[\tau]$ -algebras; and a filtered ring homomorphism $t(f^*) : H^*(Y) \rightarrow H^*(X)$.

There are motivic Milnor operations Q_k on the motivic cohomology groups [18, 13.4]:

$$Q_k : H^{i,j}(X) \rightarrow H^{i+2^{k+1}-1, j+2^k-1}(X).$$

(Note that we assume $\sqrt{-1} \in k$ so that ρ there is trivial.) Q_i are derivations on the motivic cohomology groups, $Q_i Q_j = Q_j Q_i$ if $i \neq j$ and $Q_i Q_i = 0$. They commute with pullbacks and Gysin morphisms. If x has weight $i > 0$, $Q_j x$ has weight $i - 1$ if it is not zero. These are compatible with the classical Milnor operations Q_i on $H^*(X)$ in the sense that $t(Q_i x) = Q_i t(\tau^{i-j} x)$. ([18, 3.9]). We denote it by the same notation Q_i .

1.1. Equivariant motivic cohomology. Edidin and Graham extended the construction of the Chow ring of BG to the equivariant Chow ring of a scheme with G -action, which can be generalized to motivic cohomology [2]. Let us recall the equivariant motivic cohomology [24, §8]. Suppose that G acts on a smooth quasi-affine scheme X . For each $m \geq 0$, we can choose a representation V of G with an open subscheme E on which G acts freely, and $\text{codim}_V(V - E) > m$. The quotient $(E \times X)/G$ exists as a smooth scheme. We set

$$H_G^{m,*}(X) = H^{m,*}((E \times X)/G).$$

It is independent of the choice of such V for fixed m and $*$. For a morphism of groups $\phi : H \rightarrow G$, a G -schemes X , an H -scheme Y , and an H -equivariant morphism $f : Y \rightarrow X$ where H acts on X through ϕ , there is a pull-back $f^* : H_G^{*,*}(X) \rightarrow H_H^{*,*}(Y)$. For a subgroup H of G , we see that [8, 2.1]

$$(2) \quad H_G^{*,*}((X \times G)/H) = H_H^{*,*}(X).$$

Let U be a unipotent group and consider a U -torsor $X \rightarrow Y$, we have an isomorphism $H^{*,*}(Y) = H^{*,*}(X)$, for U is isomorphic to an affine space and any U -torsor is locally trivial [5, Prop.1]. In particular for a semi-direct product $G \ltimes U$ the projection onto the component $G \ltimes U \rightarrow G$ induces an isomorphism $H^{*,*}(B(G \ltimes U)) = H^{*,*}(BG)$ [8, 2.3].

Take any two elements u_0, u_1 of $U(\mathbb{C})$. $G \rightarrow G \times U$ by $x \mapsto (x, u_i)$ induce the same morphism on $H^{*,*}(B(G \times U)) \rightarrow H^{*,*}(BG)$ for both are the section of the projection. Similarly for the multiplicative group \mathbb{G}_m : $H^{*,*}(B(G \times \mathbb{G}_m)) = H^{*,*}(BG)[y] \rightarrow H^{*,*}(BG)$. Let G be connected. For any element $g \in G(\mathbb{C})$ the conjugation by g defines a morphism $c_g : BG \rightarrow BG$, but we know that g is contained in a Borel subgroup, and $c_g^* = c_e^*$ on $H^{*,*}(BG)$ for e is the unit of G [5, Lemma.1 of Proposition 4].

1.2. The stratified method. The equivariant motivic cohomology theory has the following *localization sequence*; if we write an equivariant closed immersion $i : Y \hookrightarrow X$ of codimension s and open immersion $j : X - Y \hookrightarrow X$, there is a long exact sequence

$$\rightarrow H_G^{*-1,*}(X - Y) \xrightarrow{\delta} H_G^{*-2s,*-s}(Y) \xrightarrow{i_*} H_G^{*,*}(X) \xrightarrow{j^*} H_G^{*,*}(X - Y) \xrightarrow{\delta}$$

where i_* is the Gysin mapping.

If we know a stratification of a G -scheme X by G -subschemes, we may calculate $H_G^{*,*}(X)$ by repeated use of the localization sequence above and (1.1). This method is introduced by Vezzosi to compute $\text{CH}^*(BPGL_3)$ [17]. Molina-Rojas and Vistoli [8] calculated the Chow ring of BG using it in cases $G = GL_n$, SL_n and Sp_n over arbitrary fields, and $G = O_n$ and SO_n over fields of characteristic different from 2. The following stratification associated with O_n or SO_n are used to calculate the Chow groups and motivic cohomology in [8] and [24, §9]. Consider the standard representation of SO_n . It makes \mathbb{A}^n an SO_n -scheme. We set $B = \{x \in \mathbb{A}^n \setminus \{0\} \mid q(x) \neq 0\}$

and $C = \{x \in \mathbb{A}^n \setminus \{0\} \mid q(x) = 0\}$ where $q(x) = \sum_{k=1}^n x_k^2$. There is a stratification $\{0\} \subset \{0\} \cup C \subset \mathbb{A}^n$, and $\mathbb{A}^n \setminus (\{0\} \cup C) = B$.

Then we get two localization sequences:

$$(3) \quad \rightarrow H_{SO_n}^{*-2n, *-n}(\{0\}) \xrightarrow{s_*} H_{SO_n}^{*,*}(\mathbb{A}^n) \xrightarrow{t^*} H_{SO_n}^{*,*}(\mathbb{A}^n \setminus \{0\}) \rightarrow$$

and

$$(4) \quad \rightarrow H_{SO_n}^{*-2, *-1}(C) \xrightarrow{i_*} H_{SO_n}^{*,*}(\mathbb{A}^n \setminus \{0\}) \xrightarrow{j^*} H_{SO_n}^{*,*}(B) \xrightarrow{\delta}.$$

We have $H_{SO_n}^{*,*}(\{0\}) = H_{SO_n}^{*,*}(\mathbb{A}^n)$ and s_* is the multiplication of the n -th Chern class $c_n \in H_{SO_n}^{2n, n}(\{0\}) = H^{2n, n}(BSO_n)$ by the self-intersection formula [13]. Since in the end of proof in 3.3 we will see that $c_n \cup$ is injective, we will denote

$$(5) \quad H_{SO_n}^{*,*}(\mathbb{A}^n \setminus \{0\}) = H_{SO_n}^{*,*}/c_n.$$

We can define an action of $\mathbb{G}_m \times SO_n$ on B such that $(t, g) \in \mathbb{G}_m \times SO_n$ sends $v \in B$ to $tg(v)$, that is, the matrix multiplication of scalar $t \in \mathbb{G}_m$ and $g(v) \in B$. This action is transitive and the stabilizer of e_1 is a subgroup isomorphic to O_{n-1} . The inclusion $\kappa_1 : O_{n-1} \rightarrow \mathbb{G}_m \times SO_n$ is given by $g \mapsto (\det g, \kappa)$ where $\kappa : O_{n-1} \rightarrow SO_n$ is

$$(6) \quad g \mapsto \begin{pmatrix} \det(g) & 0 \\ 0 & g \end{pmatrix}.$$

We have $B \cong \mathbb{G}_m \times SO_n / O_{n-1}$.

Choose E as in 1.1. Then we obtain the following isomorphisms

$$\begin{aligned} H_{SO_n}^{*,*}(B) &= H^{*,*} \left(\frac{E \times (\mathbb{G}_m \times SO_n / O_{n-1})}{SO_n} \right) \\ &= H^{*,*} \left(\frac{E \times \mathbb{G}_m}{O_{n-1}} \right) = H_{O_{n-1}}^{*,*}(\mathbb{G}_m). \end{aligned}$$

\mathbb{G}_m is an O_{n-1} -scheme via \det , and $c_1(\gamma_n \otimes \mathbb{C}) = c_1(\det(\gamma_n \otimes \mathbb{C}))$, we get the identification similar to (5)

$$(7) \quad H_{O_{n-1}}^{*,*}(\mathbb{G}_m) = H_{O_{n-1}}^{*,*}/c_1$$

for we will see in the next subsection that $c_1 \cup$ is a monomorphism. Note that these are $H_{SO_n}^{*,*}$ -module isomorphism through κ^* . j^* in (4) is identified with the induced morphism of κ^* .

On the other hand, we know that SO_n transitively acts on C and the stabilizer of $e_1 + \sqrt{-1}e_2$ is isomorphic to a semi-direct product of the stabilizer of a pair $e_1 + \sqrt{-1}e_2, e_1 - \sqrt{-1}e_2$ and a unipotent subgroup, so that it is isomorphic to $SO_{n-2} \ltimes \mathbb{A}^{n-1}$ [8, 5.2]. Thus we get that $C \cong SO_n / (SO_{n-2} \ltimes \mathbb{A}^{n-1})$ and this deduces the following isomorphisms [8, 2.3]:

$$H_{SO_n}^{*,*}(C) = H_{SO_{n-2} \ltimes \mathbb{A}^{n-1}}^{*,*} = H_{SO_{n-2}}^{*,*}.$$

From the above calculations we can rewrite (4) as follows:

$$(8) \quad \rightarrow H_{SO_{n-2}}^{*-2, *-1} \xrightarrow{i_*} H_{SO_n}^{*,*}/c_n \xrightarrow{\kappa^*} H_{O_{n-1}}^{*,*}/c_1 \xrightarrow{\delta}.$$

We can regard the same stratification as that of O_n -spaces. The induced long exact sequence is exactly the one used by Yagita. We can connect two long exact sequences by the change of groups homomorphism on $\epsilon : SO_n \rightarrow O_n$:

$$(9) \quad \begin{array}{ccccccc} \rightarrow & H_{O_{n-2}}^{*-2,*-1} & \xrightarrow{(i_0)^*} & H_{O_n}^{*,*}/c_n & \xrightarrow{j_0^*} & H_{\mu_2 \times O_{n-1}}^{*,*}(\mathbb{G}_m) & \rightarrow H_{O_{n-2}}^{*-1,*-1} \rightarrow \\ & \downarrow \epsilon^* & & \downarrow \epsilon^* & & \downarrow \epsilon^* & \downarrow \epsilon^* \\ \rightarrow & H_{SO_{n-2}}^{*-2,*-1} & \xrightarrow{i_*} & H_{SO_n}^{*,*}/c_n & \xrightarrow{\kappa^*} & H_{O_{n-1}}^{*,*}(\mathbb{G}_m) & \xrightarrow{\delta} H_{SO_{n-2}}^{*-2,*-1} \rightarrow . \end{array}$$

The standard inclusion $\mathbb{A}^n \rightarrow \mathbb{A}^{n+1}$ into the first n components induces the following commutative long exact sequences (Gysin mappings and pull-backs commute in this case [20, 2.3])

$$(10) \quad \begin{array}{ccccccc} \rightarrow & H_{SO_{n-1}}^{*-2,*-1} & \xrightarrow{i_*} & H_{SO_{n+1}}^{*,*}/c_{n+1} & \xrightarrow{\kappa^*} & H_{O_n}^{*,*}/c_1 & \xrightarrow{\delta} H_{SO_{n-1}}^{*-1,*-1} \rightarrow \\ & \downarrow \iota^* & & \downarrow \iota^* & & \downarrow \iota^* & \downarrow \iota^* \\ \rightarrow & H_{SO_{n-2}}^{*-2,*-1} & \xrightarrow{i_*} & H_{SO_n}^{*,*}/c_n & \xrightarrow{\kappa^*} & H_{O_{n-1}}^{*,*}/c_1 & \xrightarrow{\delta} H_{SO_{n-2}}^{*-1,*-1} \rightarrow . \end{array}$$

Later we use another inclusion $\mathbb{A}^n \rightarrow \mathbb{A}^{n+1}$ into the second to $n+1$ -th components. It is conjugate to the composition of the standard inclusions by an element of $SO_{n+1}(\mathbb{C})$, so that the induced morphisms in motivic cohomology are the same (1.1).

1.3. $H^{*,*}(BO_n)$. It is well-known that the map induced from the restriction to the diagonal subgroup

$$H^*(BO_n) \rightarrow H^*((B\mu_2)^n)^{\Sigma_n} = \mathbb{Z}/2[x_1, \dots, x_n]^{\Sigma_n}$$

is an isomorphism, where Σ_n is the n -th symmetric group acting by permutations on x_i 's. We need three basis of the symmetric polynomials. Recall that for any partition $\lambda = [i_1, i_2, \dots, i_k, 0, \dots, 0] = [i_1, i_2, \dots, i_k, 0^{n-k}]$ of n where $\sum i_j = n$ and $i_j \neq 0$, k is called the length of λ . For each partition λ we can define a symmetric polynomial $m[\lambda] = \sum x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}$ where the sum is taken on the orbit of Σ_n . These make the monomial basis of symmetric polynomials.

The l -th Stiefel-Whitney class w_l corresponds to the elementary symmetric polynomial of degree l : $w_l = m[1, 1, \dots, 1] = m[1^l, 0^{n-l}]$, whose monomials gives a basis of $H^*(BO_n) = \mathbb{Z}/2[w_1, \dots, w_n]$.

Finally we need a basis suitable to see action of Q 's. It is due to W. S. Wilson. We denote by $Q(i)M$ the $\mathbb{Z}/2$ -module generated by $Q_{k_1} \dots Q_{k_j} m$ for all $0 \leq k_1 < \dots < k_j \leq i$ and $m \in M$. Then

$$(11) \quad H^*(BO_n) = \bigoplus_{k \geq -1} Q(k)G_k,$$

here G_{k-1} is the vector space spanned by monomial bases $m[\lambda]$ where $\lambda = [2s_1+1, 2s_2+1, \dots, 2s_k+1, 2t_1, 2t_2, \dots, 2t_q]$ for some $k+q \leq n$ ($0 \leq k, q \leq n$) and $0 \leq s_1 \leq s_2 \leq \dots \leq s_k$, $0 < t_1 \leq t_2 \leq \dots \leq t_q$ such that if the number of t equal to t_u is odd, then there exists some $v \leq k$ such that $2s_v + 2^v < 2t_u < 2s_v + 2^{v+1}$ ([21, 2.1], while the details of the combinatorial condition here are not necessary in our proof.).

Recall $Q_k x_i^{2j} = 0$ and $Q_k x_i^{2j+1} = x_i^{2j+2k}$ in $H^*((B\mu_2)^n)$. Any element of $H^*(BO_n)$ is a sum of $Q_{i_1} \dots Q_{i_s} m_\lambda$ for some such λ and $0 \leq i_1 < \dots < i_s \leq k$. Note $Q_{i_1} \dots Q_{i_s} m[\lambda] = m[2s_1 + 2i_1, 2s_2 + 2i_2, \dots, 2s_k + 2i_s, 2t_1, 2t_2, \dots, 2t_q]$, which is not zero by the condition on the partition. For example w_l is an element of the basis and $Q_0 w_l = m[2, 1^{l-1}]$, $Q_0 Q_0 w_l = m[2, 2, 1^{l-2}] + m[2, 2, 1^{l-2}] = 0$, $Q_0 Q_1 w_l = m[2, 4, 1^{l-2}]$.

Yagita determined the motivic cohomology $H^{*,*}(BO_n)$. He proved inductively that t_{BO_n} is injective by the stratified method [24, 8.2]. Then the description of the filtration by weight on $H^*(BO_n)$ directly connected to $H^{*,*}(BO_n)$ through Milnor operations (1.5). The result is the following: let $(\mu_2)^n$ be the diagonal subgroup of O_n . Then the restriction induces an isomorphism [24, 8.1]

$$(12) \quad H^{*,*}(BO_n) = (\otimes^n H^{*,*}(B\mu_2))^{\Sigma_n} = \mathbb{Z}/2[\tau] \otimes (\oplus Q(k)G_k)$$

where G_k is spanned by the monomial bases exactly the same conditions in (11) but $x_i \in H^{1,1}$.

Theorem 1.1 (Yagita [24, 8.1]). *$x \in H^*(BO_n)$ has weight k if and only if k is the maximal number such that $Q_{i_1} Q_{i_2} \dots Q_{i_k} x \neq 0$ for some tuple (i_1, i_2, \dots, i_k) .*

In particular m_λ where $\lambda = [2s_1 + 1, \dots, 2s_k + 1, 2t_1, \dots, 2t_q]$ has weight k , and x has weight 0 if and only if $Q_i x = 0$ for every i .

Corollary 1.2. *The Stiefel-Whitney class w_l in $H^*(BO_n)$ has weight l for $l = 1, \dots, n$.*

Since ι^* discards monomials of length exactly n , the following corollary is obvious.

Corollary 1.3. *The mapping induced from the canonical restriction $\iota^* : H^{*,*}(BO_n) \rightarrow H^{*,*}(BO_{n-1})$ is an epimorphism.*

2. $H^*(BSO_n)$ AND $H^{*,*}(BSO_{2m+1})$

To determine $H^{*,*}(BSO_n)$ we can not use the restriction to the diagonal subgroup. The induced mapping on motivic cohomology is neither injective nor surjective. We will use the exact sequences (3) and (8). For effective use of them we need to determine the coniveau filtration on $H^*(BSO_n)$.

2.1. Determination of weight. We will write down κ^* in terms of Stiefel-Whitney classes. Recall γ_n is the canonical n -dimensional real vector bundle

over $BO_n(\mathbb{R})$. We have the following isomorphism of vector bundles:

$$\kappa^{-1}\epsilon^{-1}\gamma_n = \det(\gamma_{n-1}) \oplus \gamma_{n-1}$$

Thus the total Stiefel-Whitney class of $\kappa^{-1}\epsilon^{-1}\gamma_n$ is

$$\begin{aligned} & 1 + w_1(\kappa^{-1}\epsilon^{-1}\gamma_n) + w_2(\kappa^{-1}\epsilon^{-1}\gamma_n) + \cdots + w_n(\kappa^{-1}\epsilon^{-1}\gamma_n) \\ &= 1 + w_1(\det \gamma_{n-1} \oplus \gamma_{n-1}) + \cdots + w_n((\det \gamma_{n-1} \oplus \gamma_{n-1})) \\ &= (1 + w_1)(1 + w_1 + \cdots + w_{n-1}) \\ (13) \quad &= 1 + (w_2 + w_1^2) + (w_3 + w_1w_2) + \cdots + (w_{n-1} + w_1w_{n-2}) + w_1w_{n-1}. \end{aligned}$$

We will prove that the mapping $t(\kappa^*): H^*(BSO_n) \rightarrow H^*(BO_{n-1})$ preserves weight strictly. We recall that a filtered mapping $\phi: H^*(X) \rightarrow H^*(Y)$ strictly preserves filtrations if $F^i H^*(X) = \phi^{-1}(F^i H^*(Y))$.

To see first $t(\kappa^*)$ is injective, we recall $\text{Ker } t(\kappa^*) \subset \text{Ker } t(\iota^*) = (w_n)$. Suppose $0 \neq x \in H_{SO_n}^*$ is in $\text{Ker } t(\kappa^*)$. It is divisible by w_n , and there exists x_j such that $x = w_n^j x_j$ and x_j is not divisible by w_n . The image $t(\kappa^*)x = w_{n-1}^j w_1^j \kappa^* x_j$ is zero, so $t(\kappa^*)x_j$ is zero, which is contradiction.

Let x be an element of $H^*(BSO_n)$ such that $t(\kappa^*)x$ is contained in the weight 0 part of $H^*(BO_{n-1})$. We can write $x = P(w_i)$ with a polynomial of w_i . $t(\kappa^*)x = P(t(\kappa^*)x)$ is a polynomial of c_i , so that each term will be a square of a monomial of w_i . By (13) it is possible, by induction on lexicographic order on monomials, only if P is a square. We have proved that $t(\kappa^*)$ is strict on the weight 0 part.

If $y = t(\kappa^*)x$ has weight $k > 0$, we switch to the motivic cohomology. Then we can find $\bar{x} \in H_{SO_n}^{*,*}$ and $\bar{y} \in H_{O_{n-1}}^{*,*}$ such that $\kappa^* \bar{x} = \tau^f \bar{y}$ for some $f \in \mathbb{Z}_{\geq 0}$. We have that

$$\kappa^* Q_{j_0} \cdots Q_{j_{k-1}} \bar{x} = \tau^f Q_{j_0} \cdots Q_{j_{k-1}} \bar{y}.$$

for some $\{j_0, \dots, j_{k-1}\} \subset \{0, \dots, n-2\}$ and $Q_{j_0} \cdots Q_{j_{k-1}} \bar{y}$ is a non-zero element in $H^{2h,h}$ for it has zero image by all Q_i . It means $f = 0$ and x has weight equal to k by the theorem of Yagita. Finally $t(\kappa^*)c_n = c_1 c_{n-1}$ implies the following.

Proposition 2.1. *The mapping $t(\kappa^*): H_{SO_n}^*/c_n \rightarrow H_{O_{n-1}}^*/c_1$ is injective and it preserves weight strictly.*

Corollary 2.2. (1) *For odd n , the Stiefel-Whitney class w_l ($2 \leq l \leq n$) in $H^*(BSO_n)$ has weight l if l is even, and $l-2$ if l is odd.*

(2) *For even n , the Stiefel-Whitney class w_n has weight $n-2$ in $H^*(BSO_n)$. For other w_l ($2 \leq l \leq n$), w_l has weight l if l is even, and $l-2$ if l is odd.*

(3) *$Q_0 w_{2l} = w_{2l+1}$ for all $1 \leq l \leq n/2$, where we regard $w_{n+1} = 0$.*

Proof. Consider $t(\kappa^*)w_l$ in $H_{O_{n-1}}^*$. If $l \neq n$, $t(\kappa^*)w_l = w_l + w_1 w_{l-1} = m[1^l] + m[1]m[1^{l-1}]$ from (13). If l is odd, $t(\kappa^*)w_l = m[2, 1^{l-1}]$. If l is even, $t(\kappa^*)w_l = w_l + w_1 w_{l-1} = m[1^l] + m[2, 1^{l-1}]$. For $l = n$, we have that $t(\kappa^*)w_n = w_1 w_{n-1} = m[2, 1^{n-2}]$. \square

2.2. The stratified method works fine in ordinary cohomology, and the result is compatible with the realization:

$$(14) \quad \rightarrow H_{SO_n}^{*-2} \xrightarrow{t(s_*)} H_{SO_n}^* \xrightarrow{t(t^*)} H_{SO_n}^*(\mathbb{A}^n \setminus \{0\}) \rightarrow .$$

We can identify as in (5) $H_{SO_n}^*(\mathbb{A}^n \setminus \{0\}) = H_{SO_n}^*/c_n$.

The sequence (8) becomes

$$(15) \quad \rightarrow H_{SO_{n-2}}^{*-2} \xrightarrow{t(i_*)} H_{SO_n}^*/c_n \xrightarrow{t(\kappa^*)} H_{O_{n-1}}/c_1 \xrightarrow{t(\delta)} H_{SO_{n-2}}^{*-1} \rightarrow$$

We have the following data on (15). These are direct consequences of Proposition 2.1, (13), and $t(\delta)$ is an $H_{SO_n}^*$ -module mapping. We may compare the result of Yagita in [24, §8] by (9). Note we will write the image in the quotient as the same letter, as is $w_i \in H_{SO_{2m}}^*/c_{2m}$.

Proposition 2.3. (1) $t(i_*) = 0$.

(2) $t(\kappa^*)w_i = w_i + w_{i-1}w_1$ for $2 \leq i \leq n-1$, and $t(\kappa^*)w_n = w_{n-1}w_1$.

(3) $t(\delta)(w_1) = 1$ and $t(\delta)(zw_1) = z$ for z is a polynomial of w_i for $2 \leq i \leq n-2$.

2.3. $H^{*,*}(BSO_{2m+1})$. Since $H_{O_n}^*$ and $H_{SO_n}^*$ are generated by the Stiefel-Whitney classes, the realization mapping $t^{*,*}$ are epimorphisms in all these cases. As in 1.3, $t: H_{O_n}^{p,q} \rightarrow H_{O_n}^p$ is injective for every pair (p, q) .

When $n = 2m+1$, we have that $O_{2m-1} \cong SO_{2m-1} \times \mu_2$. This direct product structure induces the mapping $\pi: BO_{2m-1} \rightarrow BSO_{2m-1}$ deduced from multiplication of determinant $g \mapsto \det(g)g$ if $g \in O_{2m-1}$. Then $\pi \cdot \epsilon = 1$ and $(\epsilon \cdot \pi)(\epsilon \cdot \pi) = (\epsilon \cdot \pi)$.

$$(16) \quad H_{O_{2m-1}}^{*,*}/c_1 = H_{\mathbb{Z}/2 \times SO_{2m-1}}^{*,*}(\mathbb{G}_m) = H_{SO_{2m-1}}^{*,*}\{1, x\}$$

where $x = \overline{w_1}$ has degree (1,1). It implies the realization $t: H_{SO_{2m-1}}^{p,q} \rightarrow H_{SO_{2m-1}}^p$ is injective. We can compare (8) and (15) through monomorphisms t . Because $t(\kappa^*)t_{SO_{2m+1}} = t_{O_{2m}}\kappa^*$ is injective, κ^* is injective, then we get

$$(17) \quad 0 \rightarrow H_{SO_{2m+1}}^{*,*}/c_{2m+1} \xrightarrow{\kappa^*} H_{O_{2m}}^{*,*}/c_1 \xrightarrow{\delta_2} H_{SO_{2m-1}}^{*-1,*-1} \rightarrow 0$$

3. $H^{*,*}(BSO_{2m})$

3.1. Let us denote by Y_m the kernel of $H_{SO_{2m}}^{*,*} \rightarrow H_{SO_{2m}}^*$. We will inductively determine Y_m so that we assume Y_m is known. We denote by $\overline{Y_{m+1}}$ the kernel of $t: H_{SO_{2m+2}}^{*,*}/c_{2m+2} \rightarrow H_{SO_{2m+2}}^*/c_{2m+2}$. Later we will see in 3.3 that $(\text{Ker } t_{SO_{2m}})/c_{2m+2} = \overline{Y_{m+1}}$. We can combine (8) and (15) to the

following commutative diagram

$$(18) \quad \begin{array}{ccccccc} \xrightarrow{\delta} & H_{SO_{2m}}^{*-2,*-1} & \xrightarrow{i_*} & H_{SO_{2m+2}}^{*,*}/c_{2m+2} & \xrightarrow{\kappa^*} & H_{O_{2m+1}}^{*,*}/c_1 & \xrightarrow{\delta} & H_{SO_{2m}}^{*-1,*-1} & \longrightarrow \\ & \downarrow t_{2m} & & \downarrow t_{2m+2} & & \downarrow t_{2m+1} & & \downarrow & \\ \longrightarrow & H_{SO_{2m}}^{*-2} & \xrightarrow{0} & H_{SO_{2m+2}}^{*,*}/c_{2m+2} & \xrightarrow{t(\kappa^*)} & H_{O_{2m+1}}^{*,*}/c_1 & \xrightarrow{t(\delta)} & H_{SO_{2m}}^{*-1} & \xrightarrow{0} \end{array}.$$

t_{2m+1} on the right column is injective for each degree. $\overline{w_1}$ has degree $(1, 1)$ and $t\overline{w_1} = w_1$.

We obtain that $\text{Im}(i_*) = \text{Ker}(t_{2m+2}) \subset H_{SO_{2m+2}}^{*,*}/c_{2m+2}$, using diagram-tracing argument. Note t_{2m+1} is injective. The restriction of i_* gives an isomorphism between $\text{Coker } \delta$ and $\overline{Y_{m+1}}$, and it takes Y_m into $\overline{Y_{m+1}}$. We can see $\text{Im } \delta \cap Y_m = 0$. For if there is $u = \delta v$ such that $t(u) = 0$, there exists w such that $t(\kappa^*)t(w) = t(v)$. Because $t(\kappa^*)$ strictly preserves filtrations, there is $w' \in H_{SO_{2m+2}}^{*,*}/c_{2m+2}$ such that $\kappa^*w' = v$. This means $u = 0$.

The restriction of i_* gives an acyclic subcomplex $Y_m \rightarrow Y_m$, where the second copy is in $\overline{Y_{m+1}}$, in (3.1), and we get

$$(19) \quad \longrightarrow \left(H_{SO_{2m+2}}^{*,*}/Y_m \right) / c_{2m+2} \xrightarrow{\kappa^*} H_{O_{2m+1}}^{*,*}/c_1 \xrightarrow{\delta_1} H_{SO_{2m}}^{*-1,*-1}/Y_m \longrightarrow$$

where the induced t_{2m} is now a monomorphism and $\text{Coker } \delta_1 \cong \overline{Y_{m+1}}/Y_m$.

By Proposition 2.3 we know that $\text{Coker } \delta_1$ is isomorphic to the cokernel of the restriction $\iota_{2m+1}^* : H_{SO_{2m+1}}^{*,*} \rightarrow H_{SO_{2m}}^{*,*}/Y_m$, up to permutation of coordinates, which makes a part of the following commutative exact sequences

$$(20) \quad \begin{array}{ccccccc} 0 \longrightarrow & H_{SO_{2m+1}}^{*,*}/c_{2m+1} & \xrightarrow{\kappa^*} & H_{O_{2m}}^{*,*}/c_1 & \xrightarrow{\delta_2} & H_{SO_{2m-1}}^{*-1,*-1} & \longrightarrow 0 \\ & \downarrow \iota_{2m+1}^* & & \downarrow \iota_{2m}^* & & \downarrow \iota_{2m-1}^* & \\ 0 \longrightarrow & (H_{SO_{2m}}^{*,*}/Y_m)/c_{2m} & \xrightarrow{\kappa^*} & H_{O_{2m-1}}^{*,*}/c_1 & \xrightarrow{\delta_1} & H_{SO_{2m-2}}^{*-1,*-1}/Y_{m-1} & , \end{array}$$

where ι_{2m}^* is an epimorphism.

3.2. Since $\kappa^*\overline{w_{2m}} = \overline{w_{2m}} + \tau\overline{w_1w_{2m-1}}$ in $H_{O_{2m}}^{*,*}$, $\overline{w_{2m}}$ has weight $2m$ in SO_{2m+1} . On the other hand $\overline{w_{2m}}$ in $H_{SO_{2m}}^{*,*}$ satisfies $\kappa^*\overline{w_{2m}} = \overline{w_1w_{2m-1}}$, which implies it has weight $2m-2$. That is, $\iota_{2m+1}^*\overline{w_{2m}} = \tau\overline{w_{2m}}$, and $\overline{w_{2m}}$ presents a nonzero element in $\text{Coker } \iota_{2m+1}^*$, where $\tau\overline{w_{2m}} = 0$.

We want to claim that $\overline{w_{2m}}$ is a generator of $\text{Coker } \iota_{2m+1}^*$ over the ring of even Chern classes. Since κ and ι commute, we have to consider elements having factor $\overline{w_{2m}}$ only.

Consider the long exact sequence associated with (20):

$$(21) \quad \text{Ker } \iota_{2m+1}^* \rightarrow \text{Ker } \iota_{2m}^* \rightarrow \text{Ker } \iota_{2m-1}^* \xrightarrow{\partial} \text{Coker } \iota_{2m+1}^* \rightarrow 0.$$

Connecting mapping ∂ is $(\kappa^*)^{-1} \cdot \iota_{2m}^* \cdot (\delta_2)^{-1}$. Let us take z in $\text{Ker } \iota_{2m-1}^*$. We can take $(\delta_2)^{-1}z = \overline{w_1 z}$. If it is in ordinary cohomology, $\text{Coker } \iota_{2m+1}^* = 0$, for we can take w_{2m} as $(\delta_2)^{-1}w_{2m-1}$ which go to zero by ι_{2m}^* , but in motivic cohomology $\overline{w_{2m}}$ has strictly larger weight than $\overline{w_1 w_{2m-1}}$, hence $\overline{w_{2m}}$ presents a non-zero element in $\text{Coker } \iota_{2m+1}^*$ as above.

Therefore it suffices to prove that in $H_{O_{2m}}^{*,*}$, zw_{2m} has weight less than or equal to the weight of $zw_1 w_{2m-1}$, if z is not a polynomial of even Chern classes. Let z be an element of monomial base $m[i.]$. First we assume not all i 's are even. We will use induction of length of z . If it has length 1, $m[2i+1]m[2, 1^{2m-2}, 0]$ and $m[2i+1]m[1^{2m}]$ have weight $2m-1$ ($i > 0$). $m[2i+1, 2j+1]m[2, 1^{2m-2}, 0]$ have weight more than $2m-2$ and $m[2i+1, 2j+1]m[1^{2m}]$ have weight $2m-2$. If in a monomial of length $j \geq 2$ all i are odd, zw_{2m} have weight $2m-j$, and $zw_1 w_{2m-1}$ also have weight $2m-j$. In general let $2j$ be the largest even member of i . which has multiplicity k . Let z' be the partition discarding $2j$'s from z . Then z is equal to $z'(w_k)^{2j}$ plus elements of length less than that of z . $z'(w_k)^{2j}w_{2m}$ has the same weight as that of $z'w_{2m}$, and similarly for $w_1 w_{2m-1}$. It finishes the inductive step.

If all i 's are even, it is a polynomial of $(w_i)^2$ and zw_{2m} has weight $2m$ but $zw_1 w_{2m-1}$ has weight $2m-2$. If z contains w_{2j-1}^2 , there exists u such that $zw_{2m} + uw_1 w_{2m}$ has weight $2m-2$, hence it does not represent nonzero elements in $\text{Coker } \iota_{2m+1}^*$.

3.3. We know that $Y_1 = 0$, and the cokernel of δ is easily seen to be the image of $\mathbb{Z}/2[c_2]\{\overline{w_2}\}$, here $\overline{w_2} \in H_{SO_2}^{2,1}$ in $H_{SO_4}^{*,*}/c_4$. Let us define $y_{0,2} = i_{2*}\overline{w_2}$. Then $Y_2 = \mathbb{Z}/2[c_2, c_4]\{y_{0,2}\}$ [24, §8].

Assume we know that Y_m is generated by $y_{i,m}$ over $\mathbb{Z}/2[c_2, \dots, c_{2m}]$ for $0 \leq i \leq m-2$. $i_*(y_{i,m})$ are non-zero elements in $H_{SO_{2m+2}}^{*,*}/c_{2m+2}$ (i_* is that in (3.1).) For $i_*y_{i,m}$ have degree $2m+2$ there exist $y_{i,m+1}$ in $H_{SO_{2m+2}}^{*,*}$. We will see that $(Y_{m+1}/c_{2m+2})/i_*(Y_m)$ is generated by $i_*\overline{w_{2m}}$ over $\mathbb{Z}/2[c_2, \dots, c_{2m}]$. There exists a unique element $y_{m-1,m+1}$ in $H^{2m+2,2m}(BSO_{2m+2})$ whose image by quotient of c_{2m+2} is $i_*\overline{w_{2m}}$. We get an isomorphism

$$Y_{m+1} = \mathbb{Z}/2[c_2, c_4, \dots, c_{2m+2}]\{y_{0,m+1}, \dots, y_{m-1,m+1}\}.$$

Multiplicative relations are consequence of the projection formula [13].

Finally we consider s_* in (3) as an $H_{GL_{2m}}^{*,*}$ -module mapping. $c_{2m} \cup$ is injective on $\text{Ker } \iota_{SO_{2m}}^{*,*}$ and $H_{SO_{2m}}^*$, so that we can identify $H_{SO_n}^{*,*}(\mathbb{A}^n - \{0\}) = H_{SO_n}^{*,*}/c_n$ for all n .

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